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An extremal problem on group connectivity of graphs

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ABSTRACT

Let A be an Abelian group, $n \geq 3$ be an integer, and $ex(n, A)$ be the maximum integer such that every n-vertex simple graph with at most $ex(n, A)$ edges is not A-connected. In this paper, we study $ex(n, A)$ for $|A| \geq 3$ and present lower and upper bounds for $3 \leq |A| \leq 4$ and an upper bound for $|A| \geq 5$.

1. Introduction

Graphs considered in this paper are finite, undirected, and loopless. A *simple* graph is a graph without multiple edges. The *vertex set* and *edge set* of G are denoted by $V(G)$ and $E(G)$ or simply V and E if there is no confusion. We use K_n and W_n to denote the complete graph with n vertices and the wheel on $n + 1$ vertices, respectively. For a subgraph H of G, let G/H be the subgraph obtained from G by contracting all the edges in H. An *orientation* D of a graph G is a directed graph obtained by assigning a direction to each edge in $E(G)$. For an orientation D of G and a vertex $v \in V(G)$, we use $E^+(v)$ (resp., $E^-(v)$) to denote the set of edges with tails (resp., heads) at v. Let A be an Abelian group. The *order* of A is denoted as $|A|$.

The degree of the vertex $v \in V(G)$ is the number of edges incident with it, denoted by $d_G(v)$ (or simply $d(v)$). A *k-vertex*, *$(\geq k)$-vertex* or *$(\leq k)$-vertex* is a vertex of degree k, at least k, or at most k, respectively. A k-cycle is a cycle with k vertices and a k-path is a path with k edges. Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing integer sequence. An i-element of π is a term of π whose value is i. If there are r i-elements in π, sometimes we use i^r instead of $(i, \ldots, i)$ where i repeats r times. A nonincreasing positive integer sequence $\pi = (d_1, d_2, \ldots, d_n)$ is *graphic* if there is a simple graph

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whose degree sequence is π; and such a graph is called a *realization* of π. Let $\sigma(\pi) = d_1 + d_2 + \cdots + d_n$ be the *degree sum* of π.

Let $f : E(D) \to A$ be a mapping. The *boundary* of f is the mapping $\partial f : V(D) \to A$, where $\partial f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$ for each vertex $v \in V(D)$. Note that $\sum_{v \in V(D)} \partial f(v) = 0$. We say that D is *A-connected* if, for every mapping $p : V(D) \to A$ with $\sum_{v \in V(D)} p(v) = 0$, there exists a mapping $f : E(D) \to A$ with boundary $\partial f = p$ such that f is nowhere zero, that is, $f(e) \neq 0$ for every edge $e \in E(D)$. We say that D *admits a nowhere-zero A-flow* if there is a nowhere-zero f such that ∂f is identically zero. A graphic sequence is said to *have an A-connected realization* if π has a realization that is A-connected.

As indicated in [6], the existence of a nowhere-zero mapping f with a specified boundary depends only on the underlying undirected graph and not on the orientation of the edges, since reversing an edge and negating the value f on it does not change the contribution to the boundary at the endpoints. Accordingly, we say that an undirected graph is *A-connected* (or *admits a nowhere-zero A-flow*) if it is A-connected (or admits a nowhere-zero A-flow, respectively) under some orientation.

Note that whether or not G admits a nowhere-zero A-flow only depends on the order of A. Tutte [15] proved that a graph G admits a nowhere-zero k-flow if and only if it admits a nowhere-zero A-flow for any Abelian group A with $|A| = k$. Major open problems in this area are Tutte's celebrated 3-, 4-, and 5-flow conjectures. Readers are referred to Zhang [16] for in-depth accounts.

Unlike for group flow, it is unknown whether the structure of the group A plays any role in A-connectivity. In fact, it is an open problem to determine whether any Z_4-connected graph is $Z_2 \times Z_2$-connected, or vice versa.

The concept of A-connectivity was introduced by Jaeger et al. [6] as a generalization of nowhere-zero flows. A-connected graphs are contractible configurations of A-flow and play an important role in the study of group flows because of the following: if H is A-connected, then any supergraph G of H (i.e. G containing H as a subgraph) admits a nowhere-zero A-flow if and only if G/H does.

Group connectivity is stronger than the existence of nowhere-zero flows. For example, 2-flowability is equivalent to being Eulerian, but K_1 is the only Z_2-connected graph. Nowhere-zero flows satisfy a monotonicity condition like k-colorability: if a graph admits a nowhere-zero k-flow, then it admits a nowhere-zero h-flow for any integer $h \geq k$. On the other hand, Jaeger et al. [6] showed that group connectivity is not monotone. For example, the graph consisting of four internally disjoint 3-paths with common endpoints is Z_5-connected but not Z_6-connected.

Jaeger et al. [6] generalized Seymour's 6-flow theorem and proved that every 3-edge-connected graph is A-connected with $|A| = 6$ and made the following interesting conjectures.

Conjecture 1 (*Jaeger et al. [6]*)**.**

(a) *Every 3-edge-connected graph is Z_5-connected.*
(b) *Every 5-edge-connected graph is Z_3-connected.*

Note that the conjectures above are stronger than Tutte's 5-flow/3-flow conjectures. We refer the readers to [2–4,9–12,17] for recent results on those conjectures.

A sparse graph may still admit a nowhere-zero k-flow even for $k = 2, 3, 4$—because every cycle admits a nowhere-zero 2-flow while it is not A-connected if $|A|$ is not big enough. So sufficient density is a necessary condition for a graph to be A-connected for each Abelian group with $|A| \geq 3$. This observation motivates us to study the following extremal problem for group connectivity: for an Abelian group A with $|A| \geq 3$ and an integer $n \geq 3$, find $ex(n, A)$, where $ex(n, A)$ is the maximum number such that every n-vertex simple graph with at most $ex(n, A)$ edges is not A-connected. Since the only Z_2-connected graph is K_1, there is no need to consider the case where $|A| = 2$.

Remarks. (1) For $2 \leq n < |A|$ with $|A| \geq 3$, $ex(n, A) = n - 1$. This is because the cycle with n vertices is A-connected and any simple graph with n vertices and at most $n - 1$ edges is either a tree or a disconnected graph, and neither is A-connected.

(2) By Lemma 8(a), it is easy to see that there is an A-connected graph with t edges for each integer t with $ex(n, A) + 1 \leq t \leq \frac{n(n-1)}{2}$.

In the paper, we prove the following result for $ex(n, A)$.

Theorem 2. *For an integer $n \geq 6$, $3n/2 \leq ex(n, Z_3) \leq 2n - 3$.*

Note that K_3 and K_4 are not Z_3-connected. For $n = 5$, the even wheel W_4 with five vertices (eight edges) is Z_3-connected and so is any supergraph of W_4. Clearly any graph with five vertices and at most seven edges is not Z_3-connected. Therefore, $ex(3, Z_3) = 3$, $ex(4, Z_3) = 6$, and $ex(5, Z_3) = 7$.

Theorem 3. *Let A be an Abelian group with $|A| = k \geq 5$ and $n \geq k$ be an integer with $n - 1 \equiv t \pmod{k-2}$. Then*

$$ex(n, A) < \begin{cases} \dfrac{(n-1)(k-1)}{k-2}, & \text{if } t = 0; \\ \dfrac{(n-1-t)(k-1)}{k-2} + t + 1, & \text{otherwise.} \end{cases}$$

Theorem 4. *Let A be an Abelian group with $|A| = 4$ and $n \geq 4$ be an integer. Then $(4n - 1)/3 \leq ex(n, A) \leq \lfloor (3n - 4)/2 \rfloor$.*

It is obvious that the upper bound in Theorem 4 is a special case of Theorem 3. When $|A| = 4$, we will prove a slightly more general result (Theorem 5), which concludes that any simple graph G with minimum degree at least 2 and with at least $ex(n, A) + 1$ edges either is A-connected or there is another A-connected simple graph H with the same degree sequence as G.

Theorem 5. *Let A be an Abelian group with $|A| = 4$, $n \geq 3$ be an integer, and $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 2$. If the degree sum of π, $\sigma(\pi) = d_1 + d_2 + \cdots d_n \geq 3n - 3$, then π has a realization that is A-connected. In particular, if $d_n \geq 3$, then it has an A-connected realization.*

The question of whether a degree sequence has a realization with certain properties has been extensively studied. A surprising application [14] of graph realization with 4-flows has been found in the design of critical partial Latin squares, which led to the proof of the so-called simultaneous edge-coloring conjecture of Keedwell [7] and Cameron [1]. All graphic sequences which have a realization admitting a nowhere-zero 3-flow or 4-flow are characterized in [13,14] respectively.

Note that Theorem 5 together with Conjecture 7(b) (if it is true) would characterize all graphic sequences having an A-connected realization with $|A| = 4$. We would like to propose the following conjecture.

Conjecture 6. *Let A be an Abelian group with $|A| = 4$ and $\pi = (d_1, d_2, \ldots, d_n)$ be a graphic sequence with $d_1 \geq \cdots \geq d_n \geq 2$. π has an A-connected realization if and only if the degree sum of π, $\sigma(\pi) \geq 3n - 3$.*

For the graphs constructed in the proofs of Theorems 2–4, removing one edge from each of them results in a graph which is not A-connected. So, we believe that the following is true.

Conjecture 7. *Let A be an Abelian group. Then:*

(a) $ex(n, Z_3) = 2n - 3$ *where* $n \geq 5$.
(b) $ex(n, A) = \lfloor (3n - 4)/2 \rfloor$ *where* $n \geq 3$ *and* $|A| = 4$.
(c)

$$ex(n, A) = \begin{cases} \dfrac{(n-1)(k-1)}{k-2} - 1, & \text{if } t = 0; \\ \dfrac{(n-1-t)(k-1)}{k-2} + t, & \text{otherwise.} \end{cases}$$

where $n \equiv 1 \pmod{k-2}$ *and* $|A| = k \geq 5$.

This paper is organized as follows: in Section 2, we present some useful lemmas; in Section 3, we give a proof of Theorem 2; in Section 4, we give a proof of Theorems 3 and 4; in Section 5, we prove Theorem 5.

2. Useful lemmas

Lemma 8. *Let A be an Abelian group. Then:*

(a) *(Lai [9]) A k-cycle is A-connected if and only if $|A| \geq k+1$.*
(b) *(Lai [9]) Let H be an A-connected subgraph of G. Then G is A-connected if and only if G/H is A-connected.*
(c) *(Fan et al. [4]) Every wheel with odd number of vertices is Z_3-connected.*

The following lemma provides a method for determining whether a graph is Z_3-connected: recursively remove 2-vertices; then the resulting graph is Z_3-connected if and only if the original one is.

Lemma 9 (*Zhang et al. [17]*)**.** *Let G be a graph and $v \in V(G)$ with $d_G(v) = 2$. Then G is Z_3-connected if and only if $G - v$ is Z_3-connected.*

The next corollary follows immediately from Lemma 9.

Corollary 10. *Let G be a Z_3-connected graph. Then G contains no adjacent 2-vertices.*

For A-connectivity with $|A| \geq 4$, we have the following:

Lemma 11. *Let G be a graph and A be an Abelian group with order $|A| = k \geq 4$. Let G' be a graph obtained from G and a $(k-1)$-path, say $u_0u_1 \cdots u_{k-1}$, by identifying u_0 with a vertex v_0 of G and u_{k-1} with a vertex v_{k-1} of G. Then G is A-connected if and only if G' is A-connected.*

Proof. Suppose that G is A-connected. Then G'/G is a $(k-1)$-cycle which is A-connected by Lemma 8(a). Since G is A-connected, by Lemma 8(b), G' is A-connected.

Assume that G' is A-connected; we will prove that G is A-connected as well. We first show that it is true if $v_0 = v_{k-1}$. If $v_0 = v_{k-1}$, then G' has a $(k-1)$-cycle such that we can obtain G by contracting this $(k-1)$-cycle. By Lemma 8(a) and (b), G is A-connected. In the following we assume that $v_0 \neq v_{k-1}$.

For convenience, let $A = \{a_1, a_2, \ldots, a_k\}$ with $a_k = 0$. Define $w_1 = a_1$, and for $2 \leq i \leq k-2$, let $w_i = a_i - a_{i-1}$. Note that $\sum_{i=1}^{j} w_i = a_j$ for $1 \leq j \leq k-2$ and hence $\{\sum_{i=1}^{j} w_i | 1 \leq j \leq k-2\} = A \setminus \{0, a_{k-1}\}$.

For $b \in Z(G, A)$ with $\sum_{v \in V(G)} b(v) = 0$, we define $b' \in Z(G', A)$ with $\sum_{v \in V(G')} b'(v) = 0$ as follows:

$$b'(x) = \begin{cases} b(x) & \text{if } x \in V(G) \setminus \{v_0, v_{k-1}\}, \\ b(x) + (-a_{k-1}) & \text{if } x = u_0 = v_0, \\ b(x) + a_{k-1} + (-a_{k-2}) & \text{if } x = u_{k-1} = v_{k-1}, \\ w_i & \text{if } x = u_i,\ 1 \leq i \leq k-2. \end{cases}$$

Let D be any orientation of G. Extend this orientation to an orientation of D' of G' by orienting the edge u_iu_{i+1} from u_i to u_{i+1} for $0 \leq i \leq k-2$.

Since G' is A-connected, there exists a nowhere-zero flow $f' : E(D') \mapsto A$ such that $\partial f' = b'$. Then $f'(u_ju_{j+1}) = f'(u_0u_1) + \sum_{i=1}^{j} w_i = f'(u_0u_1) + a_j \neq 0$ for $1 \leq j \leq k-2$. Then $f'(u_0u_1) \neq -a_j$ for $1 \leq j \leq k-2$. Since $a_j \neq 0$, $f'(u_0u_1) \neq 0$ and $f'(u_ju_{j+1}) = f'(u_0u_1) + a_j$ for $1 \leq j \leq k-2$, then $f'(u_su_{s+1}) \neq f'(u_tu_{t+1})$ for any $0 \leq s < t \leq k-2$. This implies that $\bigcup_{j=0}^{k-2}\{f'(u_ju_{j+1})\} = A \setminus \{0\} = \bigcup_{j=1}^{k-1}\{-a_j\}$. But $f'(u_0u_1) \neq -a_j$ for $1 \leq j \leq k-2$. Therefore $f'(u_0u_1) = -a_{k-1}$. This implies that $f'(u_{k-2}u_{k-1}) = -a_{k+1} + a_{k-2}$. Let f be the restriction of f' on D. Then $f : E(D) \mapsto A$ and $\partial f = b$. This completes the proof. □

The following lemma provides some structure for A-connected graphs with $|A| = 4$.

Lemma 12. *Let G be an A-connected simple graph with $|A| = 4$. Then either G belongs to $\mathcal{K}_3$ or G contains no vertex whose neighbors are all 2-vertices, where $\mathcal{K}_3$ is the family of graphs which consist of triangles sharing a vertex. Note that $K_3 \in \mathcal{K}_3$.*

Proof. Suppose by contradiction that G does not belong to $\mathcal{K}_3$, G is A-connected, and G contains a d-vertex v adjacent to d 2-vertices. Let $N(v) = \{v_1, \ldots, v_d\}$. We consider two cases according as

$A = Z_4$ or $A = Z_2 \times Z_2$. Since $G \notin \mathcal{K}_3$, there is at least one vertex $u \neq v$ not in $N(v)$. Hence we can always choose a boundary function $b \in Z(G, A)$ with desired values for the vertex v and its neighbors.

Case 1: $A = Z_4$.

If d is even, let $b \in Z(G, A)$ such that $b(v) = 1$ and $b(v_i) = 2$ for $1 \leq i \leq d$; if d is odd, let $b \in Z(G, A)$ such that $b(v) = 2$ and $b(v_i) = 2$ for $1 \leq i \leq d$. Then there is no nowhere-zero $f : E(G) \mapsto A$ with $\partial f = b$. Otherwise assume that there is a nowhere-zero $f : E(G) \mapsto A$ with $\partial f = b$. Since $b(v_i) = 2, f(vv_i)$ must be 1 or 3, which are odd numbers. Thus if d is even, then $\sum_{i=1}^{d} f(vv_i)$ must be even, and hence $\sum_{i=1}^{d} f(vv_i) \neq 1 = b(v)$; if d is odd, then $\sum_{i=1}^{d} f(vv_i)$ must be odd, and hence $\sum_{i=1}^{d} f(vv_i) \neq 2 = b(v)$. Therefore, G is not A-connected, a contradiction.

Case 2: $A = Z_2 \times Z_2$.

If d is even, let $b \in Z(G, A)$ such that $b(v) = (1, 1)$ and $b(v_i) = (0, 1)$ for $1 \leq i \leq d$; if d is odd, let $b \in Z(G, A)$ such that $b(v) = (0, 1)$ and $b(v_i) = (0, 1)$ for $1 \leq i \leq d$. Then there is no nowhere-zero $f : E(G) \mapsto A$ with $\partial f = b$. Otherwise assume that there is a nowhere-zero $f : E(G) \mapsto A$ with $\partial f = b$. Since $b(v_i) = (0, 1), f(vv_i)$ must be $(1, 0)$ or $(1, 1)$ which has 1 as its first component. Then if d is even, the first component of $\sum_{i=1}^{d} f(vv_i)$ must be 0, and cannot be the first component of $b(v)$ which is 1; if d is odd, then the first component of $\sum_{i=1}^{d} f(vv_i)$ must be 1, and cannot be the first component of $b(v)$ which is 0. Therefore, G is not A-connected, a contradiction. □

Let $\pi = (d_1, d_2, \ldots, d_n)$ be a nonincreasing positive integer sequence. Define $\pi' = (d_1 - 1, d_2 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1}) = (d'_1, d'_2, \ldots, d'_{n-1})$ where $d'_1 \geq d'_2 \cdots d'_{n-1}$. π' is said to be *the residual sequence obtained by laying off* d_n *from* π.

Lemma 13 (*Kleitman and Wang [8], and Hakimi [5]*)**.** *π is graphic if and only if π' is graphic.*

Lemma 14. *Let $\pi = (d_1, d_2, \ldots, d_n)$ be an integer sequence with $n - 1 \geq d_1 \geq d_2 \cdots \geq d_n \geq 2$ and $n \geq 3$. If $\sigma(\pi) = d_1 + d_2 + \cdots + d_n = 3n - 3$ is even, then π is graphic.*

Proof. Suppose by contradiction that the lemma is not true. Let $\pi = (d_1, \ldots, d_n)$ be a counterexample with n as small as possible. Let n_i be the number of i-elements of π for each $i = 2, 3$. Then $d_{n-n_2} \geq 3$ and $n_2 \geq 3$ since $d_1 + d_2 + \cdots + d_n = 3n - 3$. If $n = 3$, then $\pi = (2^3)$ which is graphic, a contradiction. Hence $n \geq 4$. Since $3n - 3$ is even, we have that n is odd and thus $n \geq 5$.

Claim 1. $d_1 \geq 4$ and $d_2 \leq n - 3$.

Proof of Claim 1. Since $n \geq 5$, we have $3n - 3 > 2n$. Hence $d_1 \geq 3$. If $d_1 = 3$, then $\pi = (3^{n_3}, 2^{n_2})$. Then $\sigma(\pi) = 3n_3 + 2n_2 = 3n - n_2$. Since $\sigma(\pi) = 3n - 3$, we have $n_2 = 3$. Note that n_3 is even. Let $n_3 = 2t$. We can construct a realization of π as follows: start with a triangle xyz; replace the edge xy with a path $xx_1x_2 \cdots x_ty$ and the edge xz with a path $xy_1y_2 \cdots y_ty$; and then add an edge x_iy_i for each $i = 1, 2, \ldots, t$. This contradicts the choice of π. This proves $d_1 \geq 4$.

Now we prove $d_2 \leq n - 3$. Otherwise assume $d_2 \geq n - 2$. Then $d_1 \geq d_2 \geq n - 2$. If $d_1 \geq n - 1$, then $\sigma(\pi) \geq d_1 + d_2 + 2(n - 2) \geq (n - 1) + (n - 2) + 2(n - 2) = 4n - 7 > 3n - 3$ since $n \geq 5$, a contradiction. Hence $d_1 = d_2 = n - 2$. Since $d_1 \geq 4$, we have $n - 2 \geq 4$ and thus $n \geq 6$. Hence $\sigma(\pi) \geq d_1 + d_2 + 2(n - 2) \geq 2(n - 2) + 2(n - 2) = 4n - 8 > 3n - 3$ since $n \geq 6$, a contradiction. This completes the proof of Claim 1. □

Consider the sequence $\pi_1 = (d_1 - 2, d_2, \ldots, d_{n-n_2}, 2^{n_2-2})$. Reorder the terms of π_1 in nonincreasing order as $\pi_1 = (d^*_1, \ldots, d^*_{n-2})$. Since, by Claim 1, $d_1 \geq 4$ and $d_2 \leq n - 3$, we have $d^*_1 \leq n - 3 = (n - 2) - 1$ and $d^*_{n-2} \geq 2$. Note that $\sigma(\pi_1) = \sigma(\pi) - 6 = 3n - 3 - 6 = 3(n - 2) - 3$. By the choice of π, π_1 is graphic. Let G be a realization of π_1 and v be a vertex in G with $d_G(v) = d_1 - 2$. Let xyz be a triangle disjoint from G. We can obtain a realization of π by identifying x with v. This contradicts the choice of π. This contradiction completes the proof of Lemma 14. □

The construction in the following lemma is crucial in the proof of Theorem 5.

Lemma 15. *Suppose that $xuvx$ is a triangle. Let t be an nonnegative integer. If we replace xu with a path $xu_1u_2 \cdots u_tu$, replace xv with a path $xv_1v_2 \cdots v_tv$ and join u_iv_i for $1 \leq i \leq t$, then the resulting graph G is A-connected with $|A| \geq 4$.*

Proof. Clearly, xu_1v_1x is a triangle in G and a triangle is A-connected. Recursively contracting a triangle, the resulting graph is K_1 which is A-connected. By Lemma 8(b), G is A-connected since a triangle is A-connected. □

3. Lower and upper bounds for $ex(n, Z_3)$

Proof of Theorem 2. We first prove the lower bound. Let G be a Z_3-connected simple graph with $n \geq 6$. Then $\delta(G) \geq 2$. By Corollary 10, G contains no adjacent 2-vertices. Thus if G has a 2-vertex, then we can recursively remove 2-vertices and the resulting graph is Z_3-connected if and only if the original one is. Note that on removing one 2-vertex, the total degree sum decreases by 4. When there is no 2-vertex, the resulting graph has minimum degree at least 3 since G is simple. Therefore the total sum of degrees of the original graph is at least $3n$ and it has at least $3n/2$ edges.

To show the upper bound, we construct a Z_3-connected graph with n vertices and $2n-2$ edges for each $n \geq 6$ as follows.

Let G be the even wheel with n vertices if n is odd and otherwise let G be the graph obtained from an even wheel with $n-1$ vertices by adding one new vertex and joining this new vertex to two vertices of the wheel. Clearly $|E(G)| = 2n-2$ and by Lemmas 8(c) and 9, G is Z_3-connected. □

Remarks. Many other Z_3-connected graphs with n vertices and exactly $2n-2$ edges can be constructed from a W_4 by recursively adding a new vertex and joining the new vertex to two other vertices already in the graph until the graph has n vertices.

4. Lower and upper bounds for $ex(n, A)$ with $|A| \geq 4$

Proof of Theorem 3. We can construct a graph G with $n \geq k-1$ vertices for $n-1 \equiv t \pmod{k-2}$ as follows: starting with a $(k-1)$-cycle, grow the graph by recursively attaching a $(k-1)$-path to the graph already obtained until the graph has $n-t$ vertices. If $t \neq 0$, attach a $(t+1)$-path to the above graph. Note that when we attach the path to the graph, the two ends of the path can be identified with the same vertex in the graph. Clearly, $|E(G)| = \frac{(n-1)(k-1)}{k-2}$ if $t = 0$ and $|E(G)| = \frac{(n-1-t)(k-1)}{k-2} + t + 1$ if $1 \leq t \leq k-3$. By Lemmas 8(a), 11, and 8(b), G is A-connected. □

Proof of Theorem 4. The upper bound is a special case of Theorem 3. We only need to prove the lower bound.

Suppose by contradiction that $ex(n, A) < (4n-1)/3$. Let G be a minimal counterexample with respect to n. Then G is A-connected and $\sum_{v \in G} d(v) \leq 2\, ex(n, A) < (8n-2)/3$.

Claim 1. G contains no adjacent 2-vertices and thus the set of all 2-vertices is an independent set of G.

Proof of Claim 1. Suppose that G has two adjacent 2-vertices v_1, v_2 such that xv_1v_2y is an induced path of G. Let $G^* = G - \{v_1, v_2\}$. Then

$$\sum_{v \in G^*} d_{G^*}(v) < \frac{8n-2}{3} - 6 < \frac{8(n-2)-2}{3} = \frac{8|V(G^*)|-2}{3}.$$

Since G is A-connected, by Lemma 11, G^* is A-connected. Therefore G^* is a smaller counterexample, a contradiction. Hence no two 2-vertices are adjacent and thus the set of all 2-vertices is an independent set. This completes the proof of Claim 1. □

Claim 2. The set of ($\geq$3)-vertices of G induces a connected subgraph G_1.

Proof of Claim 2. Otherwise, let G^* be a maximal connected subgraph of G induced by some (not all) ($\geq$3)-vertices. Contract G^* and all the resulting 2-cycles, and let v^* be the new vertex. Then we obtain a simple graph G' and all the neighbors of v^* have degree 2 by the choice of G^*. By Claim 1, $G' \notin \mathcal{K}_3$. By the definition of A-connectivity, the A-connectivity is closed under contraction. Then G' should be A-connected. But this is impossible by Lemma 12. □

Now, for any $v \in V(G) \setminus V(G_1)$, we have $d_G(v) = 2$ and its two neighbors are in $V(G_1)$. Let $n_1 = |V(G_1)|$ and n_2 be the number of 2-vertices of G. Then, $n = n_2 + n_1$.

By Claim 2, G_1 is connected. Let T_1 be a spanning tree of G_1 and $n_e = |E(G_1) \setminus E(T_1)|$. Then $n_e \geq 0$. Since each 2-vertex has two neighbors in T_1, we have

$$\begin{aligned}\sum_{v\in V(G)} d_G(v) &= \sum_{v\in V(T_1)} d_{T_1}(v) + 4n_2 + 2n_e \\ &= 2(n_1 - 1) + 4n_2 + 2n_e \\ &= 2(n_1 + n_2) + 2(n_2 + n_e) - 2 \\ &= 2n + 2(n_2 + n_e) - 2. \end{aligned} \tag{1}$$

Since $d_G(v) \geq 3$ for any $v \in V(T_1)$, we have

$$\begin{aligned} 2n_2 &\geq 3n_1 - \sum_{v\in V(T_1)} d_{T_1}(v) - 2n_e \\ &= 3n_1 - 2(n_1 - 1) - 2n_e \\ &= n_1 - 2n_e + 2 \\ &\geq n_1 - 3n_e + 2. \end{aligned} \tag{2}$$

Therefore we have $3n_2 + 3n_e \geq n_1 + n_2 + 2 = n + 2$ and thus $n_2 + n_e \geq \frac{n+2}{3}$. By Eq. (1), we have

$$\sum_{v\in V(G)} d_G(v) \geq 2n + \frac{2(n+2)}{3} - 2 = \frac{8n-2}{3},$$

a contradiction to the assumption of the theorem. This completes the proof of Theorem 4. □

5. Graphic sequences with an A-connected realization

Now we are about to prove Theorem 5.

Proof of Theorem 5. By way of a contradiction we assume that $\pi = (d_1, d_2, \ldots, d_n)$ is a smallest counterexample to the statement with respect to n. Recall that $\pi' = (d_1 - 1, d_2 - 1, \ldots, d_{d_n} - 1, d_{d_n+1}, \ldots, d_{n-1})$ is the residual sequence obtained by laying off d_n from π. Note that when $d_n \geq 2$, if π' has an A-connected realization, then π does too.

Claim 1. $d_n = 2$, $d_2 \geq 3$, and $\sigma(\pi) = 3n - 3$.

Proof of Claim 1. If $d_n \geq 4$, then each term of π' is at least 3 and hence $\sigma(\pi') \geq 3(n-1) \geq 3(n-1)-3$. If $d_n = 3$, then $\sigma(\pi) \geq nd_n \geq 3n$. Thus $\sigma(\pi') = \sigma(\pi) - 2d_n \geq 3n - 6 = 3(n-1) - 3$. Thus, if $d_n \geq 3$, then π' satisfies the assumption of the theorem. By the choice of π, π' has an A-connected realization and so has π, a contradiction. This proves $d_n = 2$.

Now we prove $d_2 \geq 3$. Otherwise assume that $d_2 = 2$. Then $\pi = (d_1, 2^{n-1})$. Since $\sigma(\pi) = d_1 + 2(n-1) \geq 3n - 3$, we have $d_1 = n - 1$ and d_1 is even. Clearly, the graph G obtained from $\frac{n-1}{2}$ triangles sharing a common vertex is a realization of π. Since each edge of G is contained in a triangle and a triangle is A-connected, G is A-connected, a contradiction.

Finally we show that $\sigma(\pi) = 3n - 3$. Otherwise assume that $\sigma(\pi) \geq 3n - 2$. Hence $\sigma(\pi') = \sigma(\pi) - 4 \geq 3n - 6 = 3(n-1) - 3$. Since $d_2 \geq 3$, π' satisfies the assumption of the theorem. By the choice of π, π' has an A-connected realization and so has π, a contradiction. Therefore $\sigma(\pi) = 3n-3$. This completes the proof of Claim 1. □

Claim 2. $d_1 \geq 4$ and $n \geq 5$.

Proof of Claim 2. By way of a contradiction we assume that $d_1 \leq 3$. By Claim 1, we have $d_1 \geq d_2 \geq 3$. Hence $d_1 = d_2 = 3$ and $\pi = (3^{n_3}, 2^{n_2})$ where n_i is the number of i-elements of π. Since $\sigma(\pi) = 3n-3$, we have $n_2 = 3$. Note that n_3 is even. Let $n_3 = 2t$. We can construct a realization of π, G, as follows: start with a triangle xyz; replace the edge xy with a path $xx_1x_2\cdots x_ty$ and the edge xz with a path $xy_1y_2\cdots y_ty$; and then add an edge x_iy_i for each $i = 1, 2, \ldots, t$. By Lemma 15, G is A-connected, a contradiction. Therefore, $d_1 \geq 4$.

Since $n \geq d_1 + 1$, we have $n \geq 5$. □

Claim 3. $d_2 \leq n-3$.

Proof of Claim 3. Suppose the contrary: that $d_2 \geq n-2$. Since $d_n \geq 2$, we have $\sigma(\pi) = 3n-3 \geq d_1+d_2+2(n-2) \geq (n-2)+(n-2)+2(n-2) = 4n-8$. Then $n \leq 5$. By Claim 2, we have $n=5$. Therefore $d_1 = n-2 = 3$, a contradiction to Claim 2. This contradiction completes the proof of Claim 3. □

The final step: Consider the sequence $\pi_1 = (d_1-2, d_2, \ldots, d_{n-n_2}, 2^{n_2-2})$. Reorder the terms of π_1 in nonincreasing order as $\pi_1 = (d_1^*, \ldots, d_{n-2}^*)$. By Claim 2 and 3, $d_1 \geq 4$ and $d_2 \leq n-3$. Thus we have $d_1^* \leq n-3 = (n-2)-1$ and $d_{n-2}^* \geq 2$. Note that $\sigma(\pi_1) = \sigma(\pi)-6 = 3n-3-6 = 3(n-2)-3$. By Lemma 14, π_1 is graphic. Since π_1 satisfies the assumption of the theorem, by the choice of π, π_1 has an A-connected realization. Let H be an A-connected realization of π_1 and v be a vertex in H with $d_H(v) = d_1-2$. Let xyz be a triangle disjoint from H. We can obtain a realization of π, denoted by G, by identifying x with v. Since H is A-connected and a triangle is A-connected, G is A-connected. This contradicts the choice of π. This contradiction completes the proof of Theorem 5. □

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